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FULLY PLASTIC CRACK IN AN INFINITE BODY UNDER ANTI-PLANE SHEAR

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Abstract—The problem of a semi-infinite body with an edge crack subjected to far out-of-plane shear is solved by a transformation to a hodograph plane and the Wiener–Hopf technique. The material stress–strain behavior is governed by a pure power hardening relation and the results are valid for both deformation theory and flow theory of plasticity. Results are presented for crack opening displacement, path independent J integral and crack tip singularities for all finite values of the power hardening parameter.

1. INTRODUCTION

The path independent J integral derived independently by Eshelby[1] and Rice[2, 3] is generally recognized as a useful parameter that characterizes the near crack tip field due to stationary cracks in elastic media. More recent studies have demonstrated that this integral provides not only an accurate characterization of the crack tip elastic–plastic field but also a good elastic–plastic fracture criterion. Noteworthy among these studies are the analytic and experimental results of Begley and Landes[4, 5] in which they propose Rice's J integral as a failure criterion. Bucci *et al.*[6] and Rice *et al.*[7] have proposed estimation procedures for J. These procedures involve the use of plastically adjusted linear elastic results in conjunction with limit load analysis. Also proposed is the estimation of J from experimentally obtained single load vs point load displacement results.

In this paper, we solve analytically the problem of a crack in an infinite body subjected to remotely applied anti-plane shear. The material stress–strain relation is governed by a power hardening law[8]—that is the normalized strain is equal to the normalized stress raised to some power. The results are therefore valid for fully plastic materials. Under the loading considered the stress history is proportional everywhere for monotonically increased loading and consequently the analysis is valid for both deformation theory and flow theory of plasticity. Results are presented for J and crack opening for the full range of the power hardening parameter that is from elastic to rigid–plastic materials. The problem for a finite strip under shear loading is under investigation, however, we note that the plane strain tensile loading of a strip has been solved by Goldman and Hutchinson[9] by the use of finite element method.

2. FUNDAMENTAL EQUATIONS AND THE HODOGRAPH PLANE

We consider the semi-infinite body occupying the region $x \geq -a$, $-\infty < y$, $z < \infty$ (see Fig. 1) with an edge crack of depth a represented geometrically by $-a < x < 0$, $y = 0$. The body is subjected to remotely applied shearing stress τ_∞. By symmetry this problem is

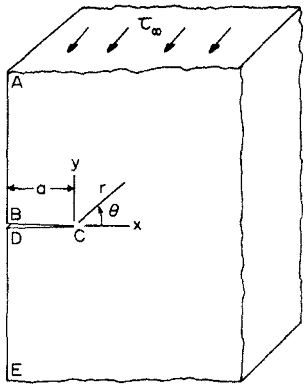

Fig. 1.

equivalent to the problem of the infinite body with a crack of width $2a$ subjected to the same remote loading. The only nonvanishing displacement component is the z component $w(x, y)$. Consequently, the nonvanishing strain components are $\gamma_{xz} = \partial w/\partial x$ and $\gamma_{yz} = \partial w/\partial y$. For small deformation and isotropic material, the corresponding stresses τ_{xz} and τ_{yz} are the only nonzero stress components. If we introduce the notation $\gamma_x = \gamma_{xz}$, $\gamma_y = \gamma_{yz}$, $\tau_x = \tau_{xz}$ and $\tau_y = \tau_{yz}$ then the compatibility and equilibrium equations reduce respectively to

$$\partial\gamma_x/\partial y = \partial\gamma_y/\partial x, \tag{1}$$

and

$$\partial\tau_x/\partial x + \partial\tau_y/\partial y = 0. \tag{2}$$

We consider a pure power hardening stress–strain relation given by

$$\gamma/\gamma_0 = \alpha(\tau/\tau_0)^n, \qquad \gamma_x/\gamma = \tau_x/\tau \quad \text{and} \quad \gamma_y/\gamma = \tau_y/\tau \tag{3}$$

where α is a nondimensional constant, γ_0 and τ_0 are reference principal strain and stress respectively, and n is the power hardening parameter. The principal stress and strain are

$$\tau = (\tau_x{}^2 + \tau_y{}^2)^{1/2}, \qquad \gamma = (\gamma_x{}^2 + \gamma_y{}^2)^{1/2}. \tag{4}$$

It is clear that because of the relation (3) the governing equation for w is also nonlinear, however, the problem can be reduced to a linear problem by the hodograph transformation. In this transformation the roles of the dependent variables (γ_x, γ_y) and independent variables (x, y) are interchanged using implicit function theory. The transformation maps the physical plane in Fig. 1 onto the strain or hodograph plane shown in Fig. 2. The hodograph transformation was used by Rice[10] to obtain a perturbation solution for the same problem for elastic–plastic materials. Details of the subsequent derivation are contained in this reference. Neuber[11] used a stress hodograph plane to analyze the doublenotched problem.

The application of this transformation to the compatability and equilibrium equations gives

$$\partial x/\partial\gamma_y = \partial y/\partial\gamma_x \tag{5}$$

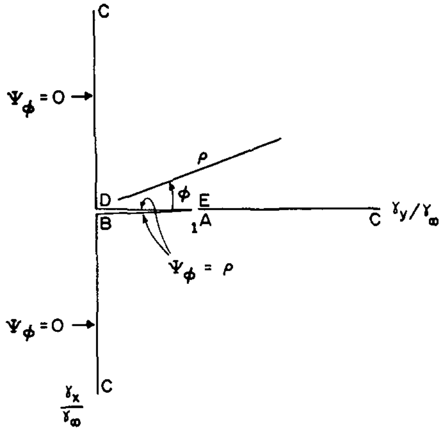

Fig. 2.

and

$$\partial x/\partial \tau_x + \partial y/\partial \tau_y = 0. \tag{6}$$

Equation (5) implies the existence of a scalar potential function ψ such that

$$\mathbf{x} = \nabla_\gamma \psi \tag{7}$$

where $\mathbf{x}$ is the position vector and ∇_γ is the gradient operator with respect to the strain vector $\gamma = (\gamma_x, \gamma_y)$. Further use of implicit function theory to relate differentiation with respect to τ_x, τ_y to differentiation with respect to γ_x, γ_y using (3) leads to the following linear partial differential equation

$$\frac{\partial^2 \psi}{\partial {\gamma_x}^2} + \frac{\partial^2 \psi}{\partial {\gamma_y}^2} + \frac{n-1}{\gamma}\left[{\gamma_x}^2 \frac{\partial^2 \psi}{\partial {\gamma_x}^2} + 2\gamma_x \gamma_y \frac{\partial^2 \psi}{\partial \gamma_x \partial \gamma_y} + {\gamma_y}^2 \frac{\partial^2 \psi}{\partial {\gamma_y}^2}\right] = 0. \tag{8}$$

One boundary condition is that $x = -a$ on AB and DE in Fig. 1 and this corresponds to

$$\partial \psi/\partial \gamma_x = -a \quad \text{for} \quad \gamma_x = 0, \qquad 0 < \gamma_y < \gamma_\infty \tag{9}$$

where γ_∞ is the corresponding remotely applied strain. The other boundary condition is that $y = 0^+$, $-a < x < 0$ on BC and DC and this leads to

$$\partial \psi/\partial \gamma_y = 0 \quad \text{for} \quad \gamma_y = 0. \tag{10}$$

It is convenient to introduce the nondimensional quantities

$$\rho = \gamma/\gamma_\infty, \qquad \Psi = \psi/a\gamma_\infty, \tag{11}$$

and a polar coordinate system (ρ, ϕ) such that

$$\gamma_x/\gamma_\infty = -\rho \sin \phi,$$

and (12)

$$\gamma_y/\gamma_\infty = \rho \cos \phi.$$

The differential equation (8) and boundary conditions (9) and (10) become

$$n\Psi,_{\rho\rho} + \frac{1}{\rho}\Psi,_{\rho} + \frac{1}{\rho^2}\Psi,_{\phi\phi} = 0, \qquad \rho > 0, \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2} \tag{13}$$

$$\Psi,_{\phi} = 0, \qquad \phi = \pm\frac{\pi}{2}, \qquad \rho > 0 \tag{14}$$

$$\Psi,_{\phi} = \rho, \qquad \phi = 0^{\pm}, \qquad 0 < \rho < 1 \tag{15}$$

where a comma subscript denotes partial derivative with respect to subsequent subscript(s).

3. WIENER–HOPF PROBLEM

The problem consisting of equations (13)–(15) can be analyzed by a method which as noted in [12] was developed by Carleman but is generally referred to as the Wiener–Hopf technique. In order to apply integral transform we need the behavior of Ψ as $\rho \to 0$ and $\rho \to \infty$.

By using equation (12) the relation (7) can be expressed in complex variable form as

$$-x/a + iy/a = \exp(-i\phi)(\psi,_{\phi}/\rho + i\psi,_{\rho}). \tag{16}$$

Now as $y \to 0^+$ and $x \to -a$

$$\Psi,_{\rho} \to \sin\phi, \tag{17}$$

and

$$\Psi,_{\phi} \to \rho\cos\phi \quad \text{as} \quad \rho \to 0$$

thus, Ψ is bounded as $\rho \to 0$.

For $\rho \to \infty$ we seek a solution to (13) and (14) of the form

$$\Psi \sim H(n, \phi)\rho^p.$$

Substituting for Ψ in equations (13) and (14) leads to

$$p = [1 - 1/n \pm \{(1 - 1/n)^2 + (2K + 1)^2/n\}^{1/2}]/2 \tag{18}$$

where K is an integer. Since Ψ must be bounded as $\rho \to \infty$, the maximum negative value of p must be chosen. This value is $p = -1/n$. Thus,

$$\Psi \to \rho^{-1/n} \quad \text{as} \quad \rho \to \infty. \tag{19}$$

We note that this leads to precisely the same singular behavior

$$\gamma \to (x^2 + y^2)^{-(n/2(n+1))}$$

near the crack tip derived by Neuber[11] and Rice[10].

Now we introduce the Mellin transform $\bar{\Psi}$ of Ψ defined by

$$\bar{\Psi}(s, \phi) = \int_0^\infty \rho^{s-1}\Psi(\rho, \phi)\,d\rho. \tag{20}$$

By (13) this transform satisfies the equation

$$\bar{\Psi},_{\phi\phi} + \omega^2(s)\bar{\Psi} = 0, \qquad 0 < Re\, s < 1/n \tag{21}$$

where

$$\omega^2(s) = s[n(s + 1) - 1], \tag{22}$$

and $Re\, s$ denotes the real part of s. The strip $0 < Re\, s < \frac{1}{n}$ of validity of the transform (21) follows from the behavior of Ψ as $\rho \to 0, \infty$ given in the relations (17) and (19). The Mellin transform of (15) is

$$\bar{\Psi}, \phi(s, 0) = \frac{1}{s+1} + \bar{u}(s), \qquad Re\, s > -1 \tag{23}$$

where $\bar{u}$ is the transform of

$$u(\rho) = \begin{cases} 0 & 0 < \rho < 1 \\ \psi, \phi(\rho, 0) & \rho > 1. \end{cases} \tag{24}$$

For $\phi > 0$ the solution of (21) and (23) is

$$\bar{\Psi}(s, \phi) = \left[\frac{1}{s+1} + \bar{u}(s)\right] \frac{\cos[\omega(s)(\phi - \pi/2)]}{\omega(s) \sin \pi\omega(s)/2}, \qquad 0 < Re\, s < 1/n \tag{25}$$

and for $\phi < 0$,

$$\bar{\Psi}(s, -\phi) = -\bar{\Psi}(s, \phi).$$

Define

$$\bar{g}(s) = \bar{\Psi}(s, 0^+) - \bar{\Psi}(s, 0^-) \tag{26}$$

where $\Psi(s, 0^+) = \lim_{\epsilon \to 0} \Psi(s, \epsilon)$ for $\epsilon > 0$. Substituting this definition into equation (25) gives

$$\tfrac{1}{2}\bar{g}(s) = \left[\frac{1}{s+1} + \bar{u}(s)\right] p(s), \qquad 0 < Re\, s < 1/n \tag{27}$$

where

$$p(s) = \omega^{-1}(s) \cot \frac{\pi}{2} \omega(s). \tag{28}$$

We note that since Ψ must be continuous across the line $\phi = 0$, $\rho > 1$ the inverse transform $g(\rho)$ of $\bar{g}(s)$ must vanish for $\rho > 1$ hence like $u(\rho)$, $g(\rho)$ is a half-known function. Furthermore, since $u(\rho) \to \rho^{-1/n}$ as $\rho \to \infty$ and $g(\rho)$ is bounded as $\rho \to 0$ then $\bar{u}(s)$ is analytic for $Re\, s < 1/n$ and $\bar{g}(s)$ is analytic for $Re\, s > 0$. Let us denote functions that are analytic in the left half plane $Re\, s < 1/n$ by a subscript $-$ and let a subscript $+$ denote functions that are analytic for $Re\, s > 0$. With this notation equation (27) becomes

$$\tfrac{1}{2}\bar{g}_+(s) = \left[\left(\frac{1}{s+1}\right)_+ + \bar{u}_-(s)\right] p(s), \qquad 0 < Re\, s < 1/n. \tag{29}$$

Equation (29) is now in the standard form for the application of the Wiener–Hopf technique (see, e.g. [12, 13]).

4. SOLUTION OF THE WIENER–HOPF EQUATION

The technique requires that $p(s)$ be decomposed into the quotient

$$p(s) = N_-(n, s)/D_+(n, s) \tag{30}$$

where $N_-(n, s)$ has no poles or zeros for $Re\, s < 1/n$ and $D_+(n, s)$ has no poles or zeros for

$Re\ s > 0$. It is noteworthy that although $\omega(s)$ has branch points $p(s)$ does not. The decomposition is readily accomplished by expressing the trigonometric functions in (28) in infinite product series.

Now, as given in [12],

$$\cos\frac{\pi}{2}\,\omega(s) = \prod_{k=1}^{\infty}\left[1 - \frac{\omega^2(s)}{(2k-1)^2}\right].$$

Unless otherwise specified k ranges over all positive integers for all subsequent product series. An explicit separation of the terms leading to zeros in the respective half planes $Re\ s > 0$ and $Re\ s < 1/n$ gives

$$\cos\frac{\pi}{2}\,\omega(s) = \Pi(\gamma_{2k-1}^{+} - a_{2k-1}s)\exp(a_{2k-1}\bar{s})$$
$$\cdot\,\Pi(a_{2k-1}s - \gamma^{-}{}_{2k-1})\exp(-a_{2k-1}\bar{s}), \tag{31}$$

where

$$\bar{s} = s + (n-1)/2n, \qquad a_m = n^{1/2}/m, \tag{32}$$

and

$$\gamma_m^{\pm} = n^{1/2}\{1/n - 1 \pm [(1 - 1/n)^2 + 4m^2/n]^{1/2}\}/2m.$$

The exponential products are introduced to render each series in (31) uniformly convergent [14]. Use has been made of the asymptotic behavior of γ_m as $m \to \infty$; namely,

$$\gamma_m^{\pm} = \pm\left[1 - \frac{n-1}{2\sqrt{n}}\cdot\frac{1}{m} + 0(m^{-2})\right]. \tag{33}$$

Thus, the desired decomposition of $p(s)$ is accomplished by setting

$$N_{-}(n, s) = B(n, s)\Pi(\gamma_{2k-1}^{+} - a_{2k-1}s)\exp(a_{2k-1}\bar{s})/\Pi(\gamma_{2k}^{+} - a_{2k}s)\exp(a_{2k}\bar{s}) \tag{34}$$

and

$$D_{+}(n, s) = \pi ns(s + 1 - 1/n)B(n, s)\Pi(a_{2k}s - \gamma_{2k}^{-})\exp(-a_{2k}\bar{s})/ 2\Pi(a_{2k-1}s - \gamma_{2k-1})\exp(-a_{2k-1}\bar{s})$$

where $B(n, s)$ is an arbitrary function which will be chosen so that N_{-} and D_{+} have algebraic behavior as $|s| \to \infty$ in the appropriate half-planes. The substitution of the quotient (30) for $p(s)$ in (29) gives

$$\tfrac{1}{2}\,\bar{g}_{+}(s)D_{+}(n, s) = \left(\frac{1}{s+1}\right)_{+} N_{-}(n, s) + \bar{u}_{-}(s)N_{-}(n, s), \qquad 0 < Re\ s < 1/n.$$

Now the first term on the right hand side of this equation can straightforwardly be decomposed into a sum. One such decomposition leads to

$$\tfrac{1}{2}\bar{g}_{+}(s)\,D_{+}(n, s) - N_{-}(n, -1)/(s+1)$$
$$= [N_{-}(n, s) - N_{-}(n, -1)]/(s+1) + \bar{u}_{-}(s)N_{-}(n, s), \qquad 0 < Re\, s < 1/n. \tag{35}$$

Since the left hand side of equation (35) is analytic, in the right half plane $Re\ s > 0$ and the right hand side is analytic in the left hand plane $Re\ s < 1/n$ and these are equal on a strip $0 < Re\ s < 1/n$ each must be an analytic continuation of the other. Thus, each side represents the same entire function $E(s)$, say.

In order to determine $E(s)$ we need the asymptotic behavior of the functions $\bar{u}_-(s)$, $\bar{g}_+(s)$, $N_-(n, s)$ and $D_+(n, s)$ as $|s| \to \infty$.

Consider the asymptotic behavior of $N_-(n, s)$ as $|s| \to \infty$, $Re\ s < 1/n$. We compare the behavior of $N_-(n, s)/B(n, s)$ as $|s| \to \infty$ with that of

$$M(s) = \Pi(1 - a_{2k-1}\bar{s})\exp(a_{2k-1}\bar{s})/\Pi(1 - a_{2k}\bar{s})\exp(-a_{2k}\bar{s}). \tag{36}$$

Now

$$\begin{aligned}\frac{N_-(n, s)}{B(n, s)M(s)} &= \prod \frac{\gamma_{2k-1} - a_{2k-1}s}{1 - a_{2k-1}\bar{s}} \prod \frac{\gamma_{2k} - a_{2k}s}{1 - a_{2k}\bar{s}} \\ &= \Pi[1 + \beta_{2k-1}(s)]\Pi[1 + \beta_{2k}(s)]\end{aligned} \tag{37}$$

where $\beta_m = (\gamma_{2k-1} - 1 + (n-1)a_{2k-1}/2n)/(1 - a_{2k-1}\bar{s})$. Thus for $Re\ s < 1/n$ and as a consequence of (33) $|\beta_m(s)| < (\text{constant})m^{-2}$. Hence each series in (37) converges uniformly and since $\lim_{s\to\infty} \beta_m(s) = 0$ then

$$\lim_{s\to\infty} \frac{N_-(n, s)}{B(n, s)M(s)} = 1 \quad \text{for} \quad Re\, s < 1/n.$$

But $M(s)$ is expressible in terms of γ functions with well-known asymptotic behavior. Thus

$$N_-(n, s) \sim (-\pi s/2)^{1/2} n^{1/4} 2^{\bar{s}\sqrt{n}} B(n, s) \quad \text{as} \quad |s| \to \infty, \qquad Re\, s < 1/n.$$

Consequently, the proper choice of $B(n, s)$ is

$$B(n, s) = 2^{-s\sqrt{n}}, \tag{38}$$

and

$$N_-(n, s) \sim (-\pi s/2)^{1/2} n^{1/4} 2^{(n-1)/2\sqrt{n}} \quad \text{as} \quad |s| \to \infty. \tag{39}$$

Similarly,

$$D_+(n, s) \sim (\pi/2)^{1/2} n^{3/4} s^{3/2} 2^{(n-1)/2\sqrt{n}} \quad \text{as} \quad |s| \to \infty, \qquad Re\, s > 0. \tag{40}$$

Now we consider the asymptotic behavior of $\bar{u}_-(s)$ and $\bar{g}_+(s)$. This behavior is dominated by the nature of $u(\rho)$ and $g(\rho)$ as $\rho \to 1^{\pm}$. This necessitates the study of Ψ in the neighborhood of $\rho = 1$. Introduce the new variables ξ and η defined by

$$\xi = n^{1/2}\gamma_x/\gamma_\infty, \qquad \eta = \gamma_y/\gamma_\infty - 1. \tag{41}$$

The partial differential equation (8) and boundary conditions (9) become

$$n\Psi_{,\xi\xi} + \Psi_{,\eta\eta} + \frac{n-1}{\frac{1}{n}\xi^2 + (\eta+1)^2}[\xi^2\Psi_{\xi\xi} + 2\xi(\eta+1)\Psi_{\xi\eta} + (\eta+1)^2\Psi_{\eta\eta}] = 0 \tag{42}$$

and

$$\Psi_{,\xi}(\xi = 0^{\pm}, \eta) = -\frac{1}{\sqrt{n}}. \tag{43}$$

It is convenient to introduce polar coordinates (r_1, β) through the relations

$$\begin{aligned} \xi &= -r_1 \sin \beta \\ \eta &= r_1 \cos \beta. \end{aligned} \tag{44}$$

For behavior of Ψ in the neighborhood of $\rho = 1$ (i.e. $r_1 = 0$), an appropriate expansion of Ψ is

$$\Psi(r_1, \beta) = g_1(\beta) r_1^{1/2} + g_2(\beta) r_1 + 0(r_1^{3/2}) \tag{45}$$

whereupon substitution into (42) and equating power of r_1 the following differential equations are obtained for g_1, g_2:

$$\begin{aligned} g'' + \tfrac{1}{4} g_1 &= 0 \\ g_2'' + g_2 &= 0 \end{aligned} \tag{46}$$

where prime denotes differentiation with respect to the argument. The corresponding boundary conditions are

$$\begin{aligned} g_1'(\pm \pi) &= 0 \\ g_2'(\pm \pi) &= -n^{-1/2}. \end{aligned} \tag{47}$$

The solutions to the boundary value problems are

$$\begin{aligned} g_1 &= c_1 \sin(\beta/2) \\ g_2 &= n^{-1/2} \sin \beta \end{aligned}$$

where c_1 is an arbitrary constant. Thus as $r_1 \to 0$

$$\Psi \sim c_1 r_1^{1/2} \sin \beta/2 + n^{-1/2} \sin \beta \gamma_1. \tag{48}$$

From this result and the relations (44), (41) and (12) it follows that

$$u(\rho) \sim (\text{constant})(\rho - 1)^{-1/2} \quad \text{as} \quad \rho \to 1^+$$

and

$$g(\rho) \sim (\text{constant})(1 - \rho)^{1/2} \quad \text{as} \quad \rho \to 1^-.$$

Hence by Watson's lemma[12]

$$\bar{u}_-(s) \to (-s)^{1/2} \quad \text{as} \quad |s| \to \infty, \qquad Re\, s < 1/n \tag{49}$$

and

$$\bar{g}_+(s) \to s^{-3/2} \quad \text{as} \quad |s| \to \infty, \qquad Re\, s > 0. \tag{50}$$

Now from the asymptotic relations (49), (50), (39) and (40) we conclude that the entire function $E(s)$ is bounded as $|s| \to \infty$, and hence by Liouville's theorem it must be a constant $K(n)$, say. Thus, from (35)

$$\frac{1}{s+1} + \bar{u}_-(s) = \frac{F(n, s)}{(s+1) N_-(n, s)} \tag{51}$$

where

$$F(n, s) = N_-(n, -1) + (s + 1) K(n). \tag{52}$$

Substituting the result (51) in equation (25) and invoking the inversion formula for the Mellin transform gives

$$\Psi(\rho, \phi) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \rho^{-s} \frac{F(n, s)\cos[\omega(s)(\phi - \pi/2)]}{(s+1)N_-(n, s)\omega(s)\sin\frac{\pi}{2}\omega(s)} \, \mathrm{d}s \qquad 0 < c < 1/n. \tag{53}$$

We now apply the theory of residues to the evaluation of this integral. The integrand has simple poles for $n \neq 1, \infty$ at $s = 0, -1 + 1/n, -1, 2m\gamma_{2m}^- n^{-1/2}$ and $(2m-1)\gamma_{2m-1}^+ n^{-1/2}$, $m = 1, 2, \ldots$. By Jordan's lemma, for $\rho > 1$ we close the contour by a large semicircle in the half plane $Re\ s > 0$ along which the integrand is small. Similarly, for $\rho < 1$ the contour is appropriately closed in the half plane $Re\ s < 0$. It follows from residue theorem that

$$\Psi(\rho, \phi) = \begin{cases} \dfrac{2F(n,0)}{\pi(n-1)N_-(n,0)} + \dfrac{2nF(n, -1+1/n)}{\pi(n-1)N_-(n, -1+1/n)} \rho^{1-1/n} + \rho \sin\phi \\ \quad + \dfrac{1}{\pi} \displaystyle\sum_{m=1}^{\infty} \frac{F(n, b_m)}{m(1+b_m)N_-(n, b_m)\omega'(b_m)} \rho^{-b_m} \cos 2m\phi, \qquad 0 < \rho < 1 \\ \displaystyle\sum_{m=1}^{\infty} \frac{2^{\sqrt{n}C_m} F(n, C_m) \prod_{k=1}^{\infty} (\gamma_{2k}^+ - a_{2k} C_m)\exp(a_{2k}\bar{C}_m)}{n^{1/2}(1+C_m) \prod_{\substack{k=1 \\ k\neq m}}^{\infty} (\gamma_{2k-1}^+ - a_{2k-1}C_m)\exp(a_{2k-1}\bar{C}_m)} \rho^{-C_m} \sin(2m+1)\phi, \qquad \rho > 1 \end{cases} \tag{54}$$

where

$$C_m = (2m-1)\gamma_{2m-1}^+ n^{-1/2}, \qquad \bar{C}_m = C_m + (n-1)/2n$$

and $b_m = 2m\gamma_m^- n^{-1/2}$ and $\gamma_m^{\pm}$ are given by equation (32). There remains only the unknown $K(n)$ contained in the expression for $F(n, s)$. This constant is determined by imposing the condition (17). For Ψ_ρ to be bounded as $\rho \to 0$ $F(n, -1 + 1/n)$ must vanish; hence

$$K(n) = -nN_-(n, -1) \tag{55}$$

and consequently,

$$F(n, s) = N_-(n, -1)[1 - n(s+1)]. \tag{56}$$

The combination of equations (54) and (16) provides a complete solution to the problem.

5. *J* INTEGRAL AND CRACK OPENING DISPLACEMENT

We compute here the path independent J integral[2] given by

$$J = \int_\Gamma \left[W\,\mathrm{d}y - \mathbf{T}\cdot\frac{\partial \mathbf{u}}{\partial x}\,\mathrm{d}s \right] \tag{57}$$

where Γ is any simple contour in the xy-plane, W is the strain energy density, $\mathbf{T}$ is the stress vector acting on the outer side of Γ and $\mathbf{u}$ is the displacement vector. For the mode of loading considered here (57) can be reduced to

$$J = -\frac{\alpha\tau_0\gamma_\infty^{1+1/n}}{(\alpha\gamma_0)^{1/n}} n\rho^{2+1/n} \frac{\partial^2}{\partial\rho^2} \int_{-\pi/2}^{\pi/2} \Psi \sin\phi \, \mathrm{d}\phi. \tag{58}$$

Now from (54)

$$\Psi \to Q(n)\rho^{-1/n}\sin\phi \quad \text{as} \quad \rho \to \infty \tag{59}$$

where

$$Q(n) = -\frac{n^{3/2}2^{\sqrt{n}}N_-(n,-1)}{n+1}\,\frac{\prod\left(\gamma_{2k}^{+} - \dfrac{1}{2k\sqrt{n}}\right)\exp((n+1)/4k\sqrt{n})}{\prod\left(\gamma_{2k+1} - \dfrac{1}{(2k+1)\sqrt{n}}\right)\exp((n+1)/2(2k+1)\sqrt{n})}. \tag{60}$$

Although the asymptotic result (59) is valid for all $n \pm \infty$, its region of validity decreases as n increases since other terms in (54) become increasingly significant in comparison to (59).

Substitution of (54) or (59) into (58) and performing the elementary manipulation gives

$$J = -\pi a\tau_\infty\gamma_\infty Q(n)(n+1)/2n. \tag{61}$$

The elimination of τ_∞ using (3) gives the equivalent result

$$J = -\pi a\tau_0\gamma_0\alpha^{-1/n}(\gamma_\infty/\gamma_0)^{(n+1)/n}Q(n)(n+1)/2n, \tag{62}$$

J is computed by evaluating the infinite product series using double precision arithmetic. The results which are accurate to five significant figures are exhibited in Table 1 and presented graphically in Fig. 3. Although we were unable to prove the following behavior of J,

$$\frac{J}{a\tau_\infty\gamma_\infty} \approx (\pi/2)^{3/2}n^{1/2} \quad \text{as} \quad n \to \infty \tag{63}$$

nevertheless because of the importance of such a formula we present it here and compare it graphically with the exact value in Fig. 3.

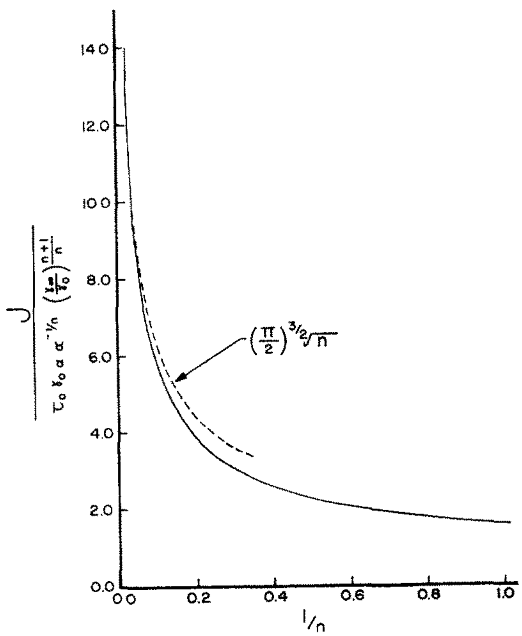

Fig. 3.

Table 1

n	$\dfrac{J}{\tau_0 \gamma_0 a \alpha^{-1/n}\left(\dfrac{\gamma_\infty}{\gamma_0}\right)\dfrac{n+1}{n}}$	$\dfrac{\delta}{a\gamma_0\left(\dfrac{\gamma_\infty}{\gamma_0}\right)}$
1·0	1·5708	2·0000
1·5	1·9389	2·3338
2·0	2·2709	2·6444
3·0	2·8638	3·2090
5·0	3·8654	4·1748
10·0	5·7878	6·0445
20·0	8·5240	8·7267
30·0	13·865	14·008
50·0	19·802	19·915
100·0	63·182	63·370

The crack opening displacement δ is defined by

$$\delta = w(x = -a, y = 0^+) - w(x = -a, y = 0^-). \tag{64}$$

From $\gamma_x = \partial w/\partial x$, $\gamma_y = \partial w/\partial y$ and the relation (7) it follows that

$$w = \boldsymbol{\gamma} \cdot \nabla_\gamma \psi - \psi + \text{const} \tag{65}$$

which is a Legendre transformation. Substituting for ψ in (65) using (54) and (11) leads to

$$\frac{\delta}{a\gamma_0\left(\dfrac{\gamma_\infty}{\gamma_0}\right)} = \frac{4N_-(n, -1)}{\pi N_-(n, 0)}. \tag{66}$$

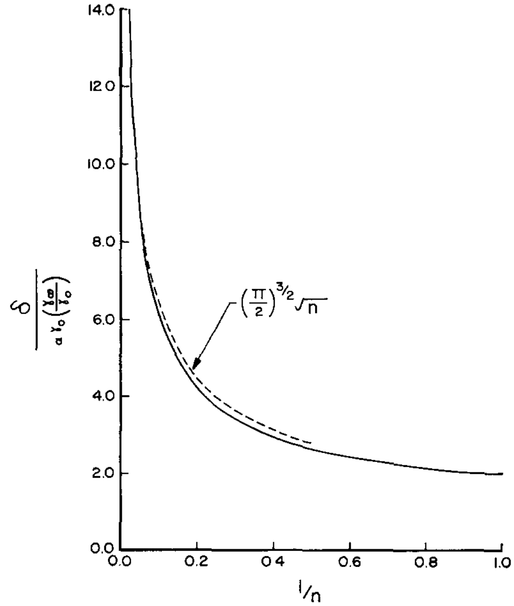

Fig. 4.

This expression was evaluated numerically and the results are given in Table 1 and displayed in Fig. 4. As for the J integral the behavior of δ as $n \to \infty$ is represented by the unproven formula

$$\frac{\delta}{a\gamma_\infty} \approx (\pi/2)^{3/2} n^{1/2}. \tag{67}$$

In Fig. 5 the dependence of $J/[\tau_0 \gamma_0 a(\delta/a\gamma_0)^{(n+1)/n}]$ on n is given. Thus, a knowledge of the power hardening parameter n and the crack opening displacement is sufficient for the determination of J integral. Such a relation may prove very useful in light of the recent experimental and analytic estimation procedures[4, 6, 7] for J.

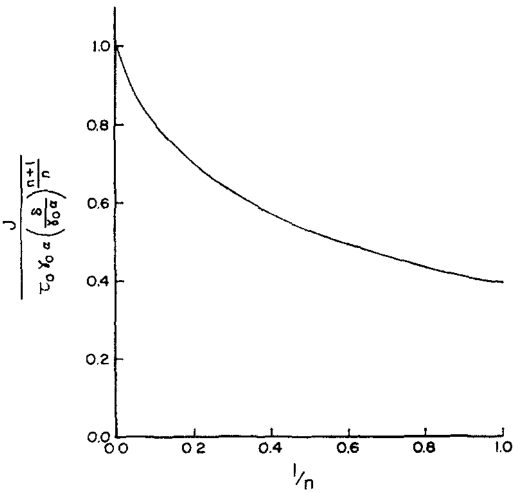

Fig. 5.

The stress, strain and displacement fields in the neighborhood of the crack can be calculated in terms of J using the asymptotic result (59). Let (r, θ) be polar coordinates centered at the crack tip in the physical xy-plane (see Fig. 1) then using a notation suggested by Rice

$$\begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \left[\frac{2J}{\pi\tau_1 r} h(\theta)\right]^{n/(n+1)} \begin{Bmatrix} -\sin\phi \\ \cos\phi \end{Bmatrix} \tag{68}$$

and

$$w = \frac{2J}{\pi\tau_1}\left[\frac{2J}{\pi\tau_1 r} h(\theta)\right]^{-1/(n+1)} \sin\phi$$

where

$$\tau_1 = \tau_0(\alpha\gamma_0)^{-1/n},$$

$$2\phi = \theta + \arcsin\left(\frac{n-1}{n+1}\sin\theta\right), \quad h(\theta) = \frac{\sin 2\phi}{2\sin\theta}.$$

The expression for the stress is omitted since it is readily obtained by using (3). The nature of the behavior in (66) was noted by Hilton and Hutchinson[15]. Here, however, the near crack tip fields are expressed in terms of Rice's J integral. For $n = 1$ the results (62), (66) and (68) reduce to the well-known results for elastic material; namely,

$$J = \pi\tau_0\gamma_0 a\alpha^{-1}(\gamma_\infty/\gamma_0)^2/2, \qquad \delta = 2a\gamma_0(\gamma_\infty/\gamma_0),$$

and

$$\begin{pmatrix}\tau_{xz}\\ \tau_{yz}\end{pmatrix} = \frac{K_{111}}{(2\pi r)^{1/2}}\begin{pmatrix}-\sin\theta/2\\ \cos\theta/2\end{pmatrix} \quad \text{with} \quad K_{111} = \tau_\infty(\pi a)^{1/2}.$$

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Абстракт — Решается задача полубесконечного тела с трещиной на краю, подверженного действию удаленного плоского сдвига, путем преобразования к плоскости годографа и методом Винера Хоифа. Определяется поведение напряжения и деформации материала посредством зависимости для чистого упрочнения мощности. Результаты важны как для теории деформации, так и для теории текучести пластичности. Даются результаты для перемещений начала трещины, независимого от траектории интеграла Райса и сингулярностей конца трещины, для всех конечных значений параметра упрочнения мощности.